If you have any questions, or notice any typo's please email Stephanie at hamilton.2016@alumni.ubc.ca or Chelsey at chelsey.hvingelby@alumni.ubc.ca

QSCU 221 Midterm 1 Review Session - Practise Questions

1. True or false:
   a. If a system of equations has 2 solutions, then it has infinitely many solutions.
   True, you can either have 0, 1 or infinitely many solutions. So 2 solutions implies you have many solutions.
   b. *If A is $m \times n$ and the equation $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^n$, then rank(A)=m
   True, "A spans $\mathbb{R}^m$"
   c. If $T: \mathbb{R}^a \to \mathbb{R}^b$ then A is an a x b matrix.
   False, A is b x a. (The domain is $\mathbb{R}^a$, *so vectors from* $\mathbb{R}^a$ *are mapped to vectors in* $\mathbb{R}^b$, meaning that the matrix must have b rows and a columns.)
   d. If A is a 4x3 matrix, then the transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^3$ onto $\mathbb{R}^4$.
   True. $\vec{x}$ *is in* $\mathbb{R}^3$ *and* $A\vec{x}$ is in $\mathbb{R}^4$.
   e. If A is a 3x2 matrix then the columns of A are linearly independent.
   False, this is not necessarily true, we would need more information about A.
   f. Let $T: \mathbb{R}^n \to \mathbb{R}^k$ be a linear transformation. Then the range of T is $\mathbb{R}^k$.
   False, $\mathbb{R}^k$ is the codomain. If T is onto, then $\mathbb{R}^k$ is the range.
   g. The matrix $\begin{bmatrix} 1 & 3 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in reduced row echelon form.
   True, there is one pivot column, and 3 free variables.
   h. If the rightmost column of a coefficient matrix is a pivot column, then the corresponding linear system is consistent.
   False, just because there is a pivot in the rightmost column does not necessarily mean that there is a solution for the linear system. Here could be a row of 0's underneath the row containing the pivot, in which case the linear system could be inconsistent.
   i. * Let $\vec{v_1}, \vec{v}_2 \in \mathbb{R}^k$. Then $\vec{0}$ is in span$\{\vec{v_1}, \vec{v}_2\}$.
   True.
   j. The columns of a 5 x 4 matrix can both span $\mathbb{R}^5$ *and be linearly independent*.
   False. The columns may be linearly independent, but they can cannot span $\mathbb{R}^5$ because # rows > # columns.
2. Consider a linear system with 231 variables and 417 equations. Let A be the coefficient matrix for the system. Determine if the following statements are **always true, sometimes true,** or **never true.** Justify your answer.
   a. $A\vec{x} = \vec{b}$ does not have a solution for all $\vec{b} \in \mathbb{R}^{417}$.

Always true. This statement translates to “the columns of A do not span $\mathbb{R}^{417}$, which is true because # rows > # columns (# equations > # variables)

b. $\vec{0}$ can be written as a non-trivial linear combination of the columns of *A*.

Sometimes true. Since # rows > # columns, the linear system may or may not have linearly independent columns. If the columns are linearly dependent then $A\vec{x} = \vec{0}$ could have a non-trivial solution. But we would need more information to know.

3. Write down the augmented matrix of the following echelon system and use the backward phase of the Gauss-Jordan algorithm to find its reduced echelon form. Then write out the solution of the system in parametric vector form.

$$x_1 + 2x_2 - 4x_3 + 2x_4 + 6x_5 = 7$$

$$-2x_3 + x_4 - 3x_5 = 9$$

$$x_4 + 3x_5 = -5$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -4 & 2 & 6 & 7 \\ 0 & 0 & -2 & 1 & -3 & 9 \\ 0 & 0 & 0 & 1 & 3 & -5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 12 & -11 \\ 0 & 0 & 1 & 0 & 3 & -7 \\ 0 & 0 & 0 & 1 & 3 & -5 \end{array}\right]$$

$x_2 = s_1$

$x_5 = s_2$

$x_4 = -5 - 3s_2$

$x_3 = -7 - 3s_2$

$x_1 = -11 - 12s_2 - 2s_1$

$$\vec{x} = \;\; = \begin{bmatrix} -11 \\ 0 \\ -7 \\ -5 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ \\ \end{bmatrix} + s_2 \begin{bmatrix} -12 \\ 0 \\ -3 \\ -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

0

0

$s_1, s_2 \in \mathbb{R}$

4. Define what it means for the vector $\vec{w}$ to be a linear combination of the vectors $\vec{v}_1, \vec{v_2}, \dots \vec{v_k}$.
Vector $\vec{w}$ is said to be a linear combination of the vectors $\vec{v}_1, \vec{v_2}, \dots \vec{v_k}$ if there exist scalars $c, c_2, \cdots, c_k$ such that $\vec{w} = c_1\vec{v_1} + C_2\vec{v}_2 + \cdots + C_k\vec{v_k}$

5. Suppose that A is a 3x3 matrix with exactly 2 pivot columns.
   a. Write down all possible reduced row echelon forms of A. (Use * to represent an entry that can have any value)

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

1 2 3

   b. If A is the coefficient matrix of a linear system, must the system be consistent? Yes or no? Why?

   NO, not every row will have a pivot position so there could be a pivot in the in the rightmost column of the corresponding augmented matrix, thus by the existence theorem the system may not be consistent.

6. Consider $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & -2 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The reduced row echelon form of $[A|b]$ is

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

   a. Express the solution to $A\vec{x} = \vec{b}$. in parametric vector form

$x_3 = s_1 \qquad x_1 = -2s_1$

$x_2 = \frac{1}{2} + s_1$

$x_4 = 0$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad s_1 \in \mathbb{R}$$

b. Can the vector $\vec{b}$ be written as a linear combination of the columns of A? If so, then find a set of weights that work. If not, explain why.

Yes!

Letting $s_1=0$:

$x_1=0$, $x_2=1/2$, $x_3=0$, $x_4=0$:

$$0\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

c. Suppose that $T(\vec{x}) = A\vec{x}$ is the linear transformation whose standard matrix is the matrix given above. What are the domain and the codomain of T?

domain $\mathbb{R}^4$ codomain: $\mathbb{R}^3$

d. Find the image of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ under T.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & -2 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$$

e. Does the vector b above have a pre-image under T? If so, find it. If not, explain why.

You haven't done this yet, so ignore this question.

7. Let $\vec{x}_1$ and $\overrightarrow{x_2}$ be solutions of $A\vec{x} = \vec{b}$. Show that $\vec{x}_1 - \vec{x}_2$ is a solution of $A\vec{x} = \vec{0}$.

1 $A\vec{x}_1 = \vec{b}$ $A\vec{x}_2 = b$ 2

$A(\vec{x}_1 - \vec{x}_2) = A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b} = \vec{0}$
1 2

8. Let A be an n x m matrix and let $\vec{b} \neq \vec{0}$ be a non-zero vector in $\mathbb{R}^n$, and suppose that vectors u, v, and w in $\mathbb{R}^m$ are any 3 solutions for the equation $A\vec{x} = \vec{b}$. Is the vector $\vec{p} = 2\vec{u} - 4\vec{v} + 3\vec{w}$ a solution of the homogeneous equation $A\vec{x} = \vec{0}$? Yes or no? Use matrix-vector algebra to justify your answer.

$$\left.\begin{aligned} A\vec{u} &= \vec{b} \\ A\vec{w} &= \vec{b} \\ A\vec{v} &= \vec{b} \end{aligned}\right\} \vec{u}, \vec{v}, \vec{w} \text{ solutions to } A\vec{x} = \vec{b}$$

$$\begin{aligned} A\vec{p} &= A(2\vec{u} - 4\vec{v} + 3\vec{w}) \\ &= A(2\vec{u}) - A(4\vec{v}) + A(3\vec{w}) \\ &= 2A(\vec{u}) - 4A(\vec{v}) + 3A(\vec{w}) \\ &= 2\vec{b} - 4\vec{b} + 3\vec{b} \\ &= \vec{b} \end{aligned}$$

$\vec{b} \neq \vec{0}$ $\therefore$ $A\vec{p} \neq \vec{0}$

so $\vec{p}$ is not a solution.

9. Let $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4]$ be a 4x4 matrix and suppose that $\vec{x} = \begin{bmatrix} 6 \\ -3 \\ 0 \\ 2 \end{bmatrix}$ is a solution of the homogeneous equation $A\vec{x} = \vec{0}$..

a. Give a linear dependence relationship on the columns of *A*.

$6\vec{a}_1 - 3\vec{a}_2 + 0\vec{a}_3 + 2\vec{a}_4 = \vec{0}$

$6\vec{a}_1 - 3\vec{a}_2 + \vec{0} + 2\vec{a}_4 = \vec{0}$

b. Do the columns of *A* span $\mathbb{R}^4$? Why or why not?

No because one of the columns (a_3) is the zero vector, so there can only be a maximum of 3 pivots, so by the existence theorem the columns of A do not span $\mathbb{R}^4$.

10. Let A be a 12x7 matrix.

a. Does the system $A\vec{x} = \vec{b}$ have a solution for every b in $\mathbb{R}^{12}$? Why or why not?

No, there are more rows than columns, so there cannot be a pivot in each row, so by the existence theorem there is not a solution for every vector b.

b. Are the columns of A linearly independent? Why or why not?

Not necessarily, we need more information. By the uniqueness theorem we need a pivot in every column, which may be possible, since there are less columns than rows, but we need more information to be sure.

11. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b+c \\ c+a \end{bmatrix}$. Find the standard matrix of *T*.

$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

12. Show that

+

a. $T:\mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = \begin{bmatrix}a\\2\end{bmatrix}$ is not linear.

$$T(\vec{x}+\vec{y}) = T\left(\begin{bmatrix}x_1+y_1\\x_2+y_2\end{bmatrix}\right) = \begin{bmatrix}x_1+y_1\\2\end{bmatrix} \neq$$

$$T(\vec{x})+T(\vec{y}) = \begin{bmatrix}x_1\\2\end{bmatrix}+\begin{bmatrix}y_1\\2\end{bmatrix} = \begin{bmatrix}x_1+y_1\\4\end{bmatrix}$$

$\therefore$ Not linear

13. Is the set $\begin{bmatrix}1\\1\\0\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}$ linearly independent or dependent? If the vectors are dependent then write down a specific linear dependence relation for them.

$$\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix} \sim \begin{bmatrix}1&0&1\\0&1&-1\\0&0&2\end{bmatrix}$$

lin independent

uniqueness thm pivot every column

14. Suppose the system below is consistent for all possible values of s and t. What can you say about the coefficients c and d? Justify your answer.

$$\begin{aligned} x_1 + 2x_2 &= s \\ cx_1 + dx_2 &= t \end{aligned}$$

$\begin{bmatrix}1\\c\end{bmatrix} \neq \#\begin{bmatrix}2\\d\end{bmatrix}$

We need to choose c and d so that the vectors $[1\ c]^T$ and $[2\ d]^T$ are linearly independent, then the system will be consistent. So as long as d does not equal 2c, we are good.

15. Determine all values of k (if any) such that the vector $\vec{b} = \begin{bmatrix}2\\k\\5\end{bmatrix}$ is in the plane given by $span\left\{\underset{v_1}{\begin{bmatrix}1\\1\\4\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}2\\3\\1\end{bmatrix}}\right\}$.

$$\begin{array}{c} \begin{matrix} v_1 & v_2 & \end{matrix} \\ \left[\begin{array}{cc|c}1&2&2\\1&3&k\\4&1&5\end{array}\right] \end{array} \sim \left[\begin{array}{cc|c}1&2&2\\0&-7&-3\\0&0&-\frac{3}{7}+k-2\end{array}\right]$$

$\underbrace{0 \quad -\tfrac{3}{7}+k-2}_{\text{needs to be}} = 0$

consistent

$$\therefore\ 0 = -\frac{3}{7} + k - 2$$

$$\Rightarrow k = 17/7$$

16. Let $T: \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation. Let $\vec{u}, \vec{v} \in \mathbb{R}^3$ be two non-zero vectors that satisfy $T(\vec{u} + 2\vec{v}) = 5, T(-\vec{u} - \vec{v}) = -5$.

   a. Use the above information to calculate $T(\vec{v})$.

$$T(\vec{u} + 2\vec{v}) + T(-\vec{u} - \vec{v}) = T(\vec{u} + 2\vec{v} - \vec{u} - \vec{v})$$
$$= T(\vec{v})$$

$$\therefore\ T(\vec{v}) = T(\vec{u} + 2\vec{v}) + T(-\vec{u} - \vec{v})$$
$$= 5 + (-5)$$
$$\boxed{T(\vec{v}) = 0}$$

   b. Explain using the definition of one-to-one why T is not a one-to-one linear transformation.

T is one to one if at most one $\vec{u} \in \mathbb{R}^3$ such that $T(\vec{u}) = \vec{b}$ for all $b \in \mathbb{R}$. Since $T(\vec{0}) = 0$ and $T(\vec{v}) = 0$, where $\vec{v} \in \mathbb{R}^3$ *is non* $-$ *zero*, there is more than one vector that maps to the zero vector.