MATH 221 Midterm 2 Practise Questions - ANSWER KEY

1. Consider a matrix equation $Ax = b$ whose corresponding augmented matrix is given by

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 0 & -3 & 9 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Write down the vector form of the general solution to **Ax=b**.

$x_1 = 1 + x_3$
$x_2 = 3 - x_3$
$x_3 = s, \quad s \in \mathbb{R}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

2. Let $[A_k] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & k^2 \end{bmatrix}$ where k is in the real numbers.

(a) Determine all values of k such that A_k is invertible.

For A_k to be invertible, the reduced row echelon form of A_k must be I_3 :

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & k^2 \end{bmatrix} \quad R_3 \leftarrow r_3 - r_2 \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & k^2 - 1 \end{bmatrix}$$

So $k^2 - 1 \neq 0 \Rightarrow k^2 = 1 \Rightarrow k = \pm 1$. For all other values of k, A_k will be invertible.

(b) Let k=0. Calculate the inverse of A_k.

$A_0 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Find $\begin{bmatrix} A_0 | I_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$. So $A_k^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

(c) Calculate the solutions to this linear system using the inverse you found in (b):

$$x_1 - x_2 = -3$$
$$x_2 + x_3 = 1$$
$$x_2 = 0$$

The linear system written as a matrix equation is:

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. A is the same as A_0 from part (b), so $A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$. So multiplying both side of the matrix equation by A^{-1} we get the following:

$$A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b} \Rightarrow I\boldsymbol{x} = A^{-1}\boldsymbol{b} \Rightarrow \boldsymbol{x} = A^{-1}\boldsymbol{b}$$

$$x = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

3. Use the row reduction method to find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ if it is invertible, and state your answer clearly. If it is not invertible, state a clear reason why it is not.

Use row reduction:

$\left[\begin{array}{ccc|ccc} 2 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$. Since we get the identity on the right hand side (right 3 columns), then A is invertible, and

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Find the inverse of $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ using the 2 by 2 inversion formula.

The formula is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So we have: $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \frac{1}{3\cdot 2 - 5\cdot 1}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

5. Let A = $\begin{bmatrix} 1 & -1 & -2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$, C = $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Determine which of the following 4 expressions are defined. If they are defined, then calculate the expression. If they are not defined, then give a reason why.

(a) $A^T B$

Defined.

$$A^T = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & -2 \\ 1 & 2 \end{bmatrix}; A^T B = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ -2 & 5 \\ 2 & -3 \end{bmatrix}$$

(b) $C^{-1}A$

Not defined. C^{-1} does not exist because $det(C) = 0\cdot 1 - 0\cdot 0 = 0$

(c) $B(B^T)^{-1} + I_2$

$$B^T = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}; (B^T)^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}; B(B^T)^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$$

$$B(B^T)^{-1} + I_2 = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}$$

(d) $CB^{-1} + I_3$

Not defined. CB^{-1} produces a 2 x 2 matrix, but I_3 is 3 x 3, so they cannot be added together.

6. A matrix is symmetric if it equals its own transpose. Use algebra properties to show that AA^T is symmetric for all n x m matrics A.

To be symmetric the following needs to be true: $(AA^T)^T = AA^T$

$(AA^T)^T = (A^T)^T A^T = AA^T$, therefore AA^T is symmetric.

7. Calculate the matrix $AA^T - 6\,I_2$, where $A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 1 & 3 \end{bmatrix}$. Show your steps.

$$\begin{bmatrix} 3 & -1 & 2 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 1 \\ 2 & 3 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 2 \\ 2 & 11 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$$

8. (a) Suppose that A, B, X are n x n matrices and assume that A and X are invertible. Solve the equation $(XA)^{-1}(A + X) = B$ for the matrix X in terms of A and B. Show your steps. State clearly which matrix you need to assume is invertible.

$(XA)^{-1}(A + X) = B$

$(XA)\cdot(XA)^{-1}(A + X) = (XA)B$ (XA) is invertible because it is the product of two invertible matrices

$I(A + X) = XAB$

$A + X = XAB$

$A = XAB - X$ move X to one side

$A = X(AB - I)$

$A(AB - I)^{-1} = X(AB - I)\cdot(AB - I)^{-1}$ * assuming $AB - I$ is invertible

$A(AB - I)^{-1} = X(I)$

$A(AB - I)^{-1} = X$

(b) Prove that what you assumed to be invertible in part (a) is invertible.

In part (a) we assumed that $AB - I$ is invertible.

$AB - I$ is invertible because in part (a) we saw that $A = X(AB - I) \Rightarrow (AB - I) = X^{-1}A$, so $(AB - I)$ is the product of two invertible matrices, therefore it is also invertible.

9. (a) Let $S \subseteq \mathbb{R}^n$ be a subspace. State the definitions of a basis β for S and of the dimension of S.

A basis for S is a linearly independent set of vectors that spans S.

The dimension fo S si the number of vectors in a basis for S.

(b) Suppose that $S' \subseteq \mathbb{R}^4$ is a subspace that contains 3 linearly independent vectors v_1, v_2, v_3. Why is it not necessarily true that these three linearly independent vectors are a basis for S'?

Since these are linearly independent vectors in S, either they are a basis for S or a finite number of vectors can be added to the set to form a basis. Since $S' \subseteq \mathbb{R}^4$, dim($S$) could be 4, so another vector would have to be added to the set in order to form a basis. So v_1, v_2, v_3 are not necessarily a basis for S.

(c) If $\boldsymbol{v} \in \mathbb{R}^4$ but $\boldsymbol{v} \notin S'$, must v_1, v_2, v_3 necessarily be a basis for S'? Why?

Yes, if $v \in \mathbb{R}^4$ but $v \notin S'$, then the basis of S does not span $\mathbb{R}^4$ so $\dim(S) < 4$. If $\dim(S') < 4$ and v_1, v_2, v_3 are linearly independent vectors in S', then they must form a basis.

10. Consider the following matrix A whose reduced row echelon form is given

$$A = \begin{bmatrix} -3 & 6 & -2 & 4 & -5 & 0 \\ 2 & -4 & 1 & -3 & 4 & -1 \\ 1 & -2 & 4 & 2 & -5 & -1 \\ 6 & -12 & 1 & -11 & 16 & 40 \\ 4 & -8 & 2 & -6 & 8 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Fill in the blanks:

(i) Col(A) is a subspace of $\mathbb{R}^5$.

(ii) Null(A) is a subspace of $\mathbb{R}^6$.

(iii) Row(A) is a subspace of $\mathbb{R}^6$.

(b) What is rank(A), dim(Null(A)), dim(Row(A))?

rank(A) = dim(Col(A)) = 3

dim(Null(A)) = n - rank(A) = 6 - 3 = 3

dim(Row(A)) = rank(A) = 3

(c) Write down bases for Col(A), Null(A), and Row(A).

basis for Col(A) are the columns that correspond to pivot columns in the echelon form of A:

$$\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 40 \\ 1 \end{bmatrix} \right\}$$

basis for Null(A):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = s \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \cdot \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$ where $s, t, u \in \mathbb{R}$, so a basis for Null(A) is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

basis for Row(A): pivot rows in echelon form of A

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

11. Consider $B = \begin{bmatrix} 6 & -4 & 2 & 3 & 3 \\ -3 & 2 & -1 & 0 & -3 \\ 12 & -8 & 4 & 7 & 5 \\ 9 & -6 & 3 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{3} & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Use this to answer the following questions. Fill in the blanks for part (b).

(a) What is a basis for Col(B)?

Remember to use the columns in the original B that correspond to pivot columns in the reduced row echelong form of B.

$$\left\{ \begin{bmatrix} 6 \\ -3 \\ 12 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 7 \\ 2 \end{bmatrix} \right\}.$$

(b) Col(B) is 2 - dimensional subspace of $\mathbb{R}^4$.

(c) Is the vector in $\begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in Null(B)? Why or why not?

This is asking if $A\boldsymbol{x} = \boldsymbol{b}$.

$$\begin{bmatrix} 6 & -4 & 2 & 3 & 3 \\ -3 & 2 & -1 & 0 & -3 \\ 12 & -8 & 4 & 7 & 5 \\ 9 & -6 & 3 & 2 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So yes, this vector is in Null(B).

(d) What is a basis for Null(B)?

Write the reduced row echelon frm of B in vector form:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s_1 \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + s_2 \cdot \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_3 \cdot \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, s_1, s_2, s_3 \in \mathbb{R}$$

So a basis for Null(B) is

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

12. Let A be a 15 x 18 matrix. Suppose that the reduced row echelon form of A^T has 12 rows of zeros.

(a) What is rank(A) and dim(Null(A))? Justify your answer. (Hint: think about how rank(A) and rank(A^T) are related. How is rank(A) related to dim(Row(A))?)

Note that $rank(A) = rank(A^T)$. 12 rows of zeros implies that 6 rows are non-zero, since A^T is 18 x 15. Each non-zero row has a pivot row, so $rank(A^T) = 6$, thus $rank(A) = 6$.

By the rank-nullity theorem*: $dim(Null(A)) = n - rank(A) = 18 - 6 = 12.$

(b) How many solutions does $A\mathbf{x} = \mathbf{0}$ have? Why?

There are three non-zero basis vectors in the null space of B, so any combination of these three vectors will satisfy the equation $A\mathbf{x} = \mathbf{0}$, so there are infinitely many solutions to $A\mathbf{x} = \mathbf{0}$.

13. Let $S \subseteq \mathbb{R}^n$ be a non-zero subspace with dim(S) = m. Let $\beta = \{v_1, ..., v_m\}$ be a set of m vectors in S. Prove that if β spans S, then β is a basis for S (this is the second half of The Basis Theorem*).

If B spans S, then either B is a basis for S or a finite number of vectors can be removed from B to form a basis for S (*question: which theorem states this?). However if vectors are removed from B, then B would contain less than m vectors, but dim(S)=m, so a basis has to contain m vectors. Thus, B is a basis for S.

14. Suppose that S is a subspace of $\mathbb{R}^3$ and that it contains the two independent vectors $\boldsymbol{a} = \begin{bmatrix} 6 \\ 1 \\ -4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 5 \\ 8 \end{bmatrix}$. Find another non-zero vector in S.

Any linear combination of $\boldsymbol{a}$ and $\boldsymbol{b}$ will be in S, ex:

$$0 \cdot a + 2 \cdot b = 2 \cdot \begin{bmatrix} 0 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 16 \end{bmatrix}$$

15. Answer the questions true or false:

(a) Let A and B be two matrices of the same size and such that $A^T = -A$. Then $(A+B)^T = A + B^T$.

FALSE

(b) Let A be a 35 x 25 matrix. Then rank(A) can be equal to 35.

FALSE

(c) $\mathbb{R}^2$ has infinitely many subspaces of dimension 1.

TRUE

(d) Let A be an n x n matrix and let $\boldsymbol{b} \in \mathbb{R}^n$. Then the set of all solutions to $A\boldsymbol{x} = \boldsymbol{b}$ form a subspace of $\mathbb{R}^n$.

FALSE

(e) Let A be 3 x 4 and let B be 4 x 3. Then $(AB)^{-1} = B^{-1}A^{-1}$.

FALSE