MATH 221 Midterm 2 Practise Questions
(This is not the same length as a midterm, just some midterm like questions. Some questions may be similar ot others. Do not expect that to happen on a test.)

1. Consider a matrix equation $A\mathbf{x} = \boldsymbol{b}$ whose corresponding augmented matrix is given by

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Write down the vector form of the general solution to **Ax=b**.

2. Let $[A_k] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & k^2 \end{bmatrix}$ where k is in the real numbers.

(a) Determine all values of k such that A_k is invertible.

(b) Let k=0. Calculate the inverse of A_k.

(c) Calculate the solutions to this linear system using the inverse you found in (b):

$$x_1 - x_2 = -3$$
$$x_2 + x_3 = 1$$
$$x_2 = 0$$

3. Use the row reduction method to find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ if it is invertible, and state your answer clearly. If it is not invertible, state a clear reason why it is not.

4. Find the inverse of $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ using the 2 by 2 inversion formula.

5. Let A = $\begin{bmatrix} 1 & -1 & -2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$, C = $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Determine which of the following 4 expressions are defined. If they are defined, then calculate the expression. If they are not defined, then give a reason why.

(a) $A^T B$

(b) $C^{-1} A$

(c) $B(B^T)^{-1} + I_2$

(d) $CB^{-1} + I_3$

6. A matrix is symmetric if it equals its own transpose. Use algebra properties to show that AA^T is symmetric for all n x m matrics A.

7. Calculate the matrix $AA^T - 6\, I_2$, where $A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 1 & 3 \end{bmatrix}$. Show your steps.

8. (a) Suppose that A, B, X are n x n matrices and assume that A and X are invertible. Solve the equation $(XA)^{-1}(A + X) = B$ for the matrix X in terms of A and B. Show your steps. State clearly which matrix you need to assume is invertible.

(b) Prove that what you assumed to be invertible in part (a) is invertible.

9.
(a) Let $S \subseteq \mathbb{R}^n$ be a subspace. State the definitions of a basis β for S and of the dimension of S.

(b) Suppose that $S' \subseteq \mathbb{R}^4$ is a subspace that contains 3 linearly independent vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$. Why is it not necessarily true that these three linearly independent vectors are a basis for S'?

(c) If $\boldsymbol{v} \in \mathbb{R}^4$ but $\boldsymbol{v} \notin S'$, must $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ necessarily be a basis for S'? Why?

10. Consider the following matrix A whose reduced row echelon form is given

$$A = \begin{bmatrix} -3 & 6 & -2 & 4 & -5 & 0 \\ 2 & -4 & 1 & -3 & 4 & -1 \\ 1 & -2 & 4 & 2 & -5 & -1 \\ 6 & -12 & 1 & -11 & 16 & 40 \\ 4 & -8 & 2 & -6 & 8 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Fill in the blanks:
  (i) Col(A) is a subspace of _______________.

  (ii) Null(A) is a subspace of _______________.

  (iii) Row(A) is a subspace of _______________.

(b) What is rank(A), dim(Null(A)), dim(Row(A))?

(c) Write down bases for Col(A), Null(A), and Row(A).

11. Consider

$$B=\begin{bmatrix} 6 & -4 & 2 & 3 & 3 \\ -3 & 2 & -1 & 0 & -3 \\ 12 & -8 & 4 & 7 & 5 \\ 9 & -6 & 3 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{3} & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

. Use this to answer the following questions. Fill in the blanks for part (b).

(a) What is a basis for Col(B)?

(b) Col(B) is _____ - dimensional subspace of $\mathbb{R}^{-}$.

(c) Is the vector in $\begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in Null(B)? Why or why not?

(d) What is a basis for Null(B)?

12. Let A be a 15 x 18 matrix. Suppose that the reduced row echelon form of A^T has 12 rows.

(a) What is rank(A) and dim(Null(A))? Justify your answer. (Hint: think about how rank(A) and rank(A^T) are related. How is rank(A) related to dim(Row(A))?)

(b) How many solutions does $A\boldsymbol{x} = \boldsymbol{0}$ have? Why?

13. Let $S \subseteq \mathbb{R}^n$ be a non-zero subspace with $\dim(S) = m$. Let $\beta = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ be a set of m vectors in S. Prove that if β spans S, then β is a basis for S (this is the second half of The Basis Theorem*).

14. Suppose that S is a subspace of $\mathbb{R}^3$ and that it contains the two independent vectors $\boldsymbol{a} = \begin{bmatrix} 6 \\ 1 \\ -4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 5 \\ 8 \end{bmatrix}$. Find another non-zero vector in S.

15. Answer the questions true or false:

(a) Let A and B be two matrices of the same size and such that $A^T = -A$. Then $(A + B)^T = A + B^T$.

(b) Let A be a 35 x 25 matrix. Then rank(A) can be equal to 35.

(c) $\mathbb{R}^2$ has infinitely many subspaces of dimension 1.

(d) Let A be an n x n matrix and let $\boldsymbol{b} \in \mathbb{R}^n$. Then the set of all solutions to $A\boldsymbol{x} = \boldsymbol{b}$ form a subspace of $\mathbb{R}^n$.

(e) Let A be 3 x 4 and let B be 4 x 3. Then $(AB)^{-1} = B^{-1}A^{-1}$.