MATH 221

QSCU Midterm Review Session

qscu.org

The Quantitative Sciences Course Union (QSCU)

Introductions

Name, Major, Role in QSCU

The Quantitative Sciences Course Union (QSCU)

How do I get involved?

Weekly Meetings:

Tuesdays, 3:30 - 5:00

UNC Boardroom

The Quantitative Sciences Course Union (QSCU)

How do I stay in the loop?

Facebook.com/ubcoqscu

QSCU.org

Overview of Topics

- Matrix Operations
- The Inverse of a Matrix
- Characteristics of Invertible Matrices
- Subspaces of Rn
- Dimension and Rank
- Introduction to Determinants
- Properties of Determinants

Matrix Operations

Matrices

- What does a diagonal matrix look like?
- What does the identity matrix look like?
  - Is the identity matrix a diagonal matrix?
- What does the zero matrix look like?
- What does it mean for two matrices to be **equal**?

Matrix Operations and Scalar Multiples

- Note that the sum A + B is defined only when A and B are the same size!!!!!!
- If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose columns are r times the corresponding columns in A.

Theorem 1

- Let A, B, and C be matrices of the same size, and let r and s be scalars.
  - $A + B = B + A$
  - $(A + B) + C = A + (B + C)$
  - $A + 0 = A$
  - $r(A + B) = rA + rB$
  - $(r+s)A = rA + sA$
  - $r(sA) = (rs)A$

Properties of Matrix Multiplication: Theorem 2

- Recall matrix multiplication
- What about squaring a matrix? What do we do?
- Let A be an mxn matrix, and let B and C have sizes for which the indicated sums and products are defined.
  - $A(BC) = (AB)C$ (associative law of multiplication)
  - $A(B + C) = AB + AC$ (left distributive law)
  - $(B + C)A = BA + CA$ (right distributive law)
  - $r(AB) = (rA)B = A(rB)$ for any scalar r
  - $I_mA = A = AI_n$ (identity for matrix multiplication)

Some Warnings!!!!

1) In general, AB does not equal BA
2) The cancellation laws do not hold for matrix multiplication (you can not cancel across the equals sign)
3) If a product AB is the zero matrix, you *cannot* conclude that in general A = 0 or B = 0.

Transpose of a Matrix

- Given an mxn matrix A, the transpose of A is the nxm matrix, denoted by A^T, whose columns are formed from the corresponding rows of A
  - Swapping the rows and columns in a sense
- The transpose of a product of matrices equals the product of their transposes in the reverse order.
  - $(AB)^T = B^T A^T$

Theorem 3

- Let A and B denote matrices whose sizes are appropriate for the following sums and products
  - $(A^T)^T = A$
  - $(A + B)^T = A^T + B^T$
  - For any scalar r, $(rA)^T = rA^T$
  - $(AB)^T = B^TA^T$

The Inverse of a Matrix

Matrix Operations

- An nxn matrix A is said to be **invertible** if there is an nxn matrix C such that
    - CA = I and AC = I

Where I = I_n, then nxn identity matrix

- In this case, C is an **inverse** of A
- $A^{-1}A = I$ and $AA^{-1} = I$
    - It does not matter if you left multiply or right multiply by the inverse -- but whichever you choose, be consistent if you are multiplying an entire equation

Matrix Operations: Theorem 4

- **Theorem 4:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then

  A is invertible and

  $$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  If $ad - bc = 0$, then A is not invertible.
- The quantity $ad - bc$ is called the **determinant** of A, and we write $\det A = ad - bc$
- This theorem says that a 2×2 matrix A is invertible if and only if $\det A \neq 0$

Example!

Theorem 5

- **Theorem 5**: If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

- This is really just saying we can solve the above equation. Let's see how this would be done...

Theorem 6

- **Theorem 6:**

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

Elementary Matrices

- An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix

▪ **Theorem 7**: An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1} .

- We will see an algorithm for solving for the inverse matrix on the next slide...

Algorithm for Finding A^-1

- Recall theorem 7, this is the underlying principle behind the algorithm
- Best shown through an example!
- How could you check your answer to make sure it is correct?

Characterizations of Invertible Matrices

Theorem 8 - The Invertible Matrix Theorem

- **Theorem 8**: Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.
  a. A is an invertible matrix.
  b. A is row equivalent to the $n \times n$ identity matrix.
  c. A has n pivot positions.
  d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  e. The columns of A form a linearly independent set.

Theorem 8 - The Invertible Matrix Theorem

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

Theorem 9

- **Theorem 9**: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying equation (1) and (2).

Some Comments on the Invertible Matrix Theorem

- Each statement in the theorem describes a property of every nxn invertible matrix
- The negation of a statement in the theorem describes a property of every nxn singular matrix (non invertible matrix)
- For instance, an nxn singular matrix is not row equivalent to I_n, does not have n pivot positions, and has linearly dependent columns.
- The Invertible Matrix Theorem *applies only to square matrices*

Invertible Linear Transformations

- A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \qquad (1)$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \qquad (2)$$

Subspaces of Rn

Subspaces of R^n

- **Definition**: A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

  a) The zero vector is in H.

  b) For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

  c) For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Column Space and Null Space of a Matrix

- **Definition:** The **column space** of a matrix A is the set Col A of all linear combinations of the columns of A.
- If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$ with the columns of $\mathbb{R}^n$, then Col A is the same as Span$\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Example 4 shows that the column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

Column Space and Null Space of a Matrix

- **Definition:** The **null space** of a matrix A is the set Nul A of all solutions of the homogenous equation $A\mathbf{x} = \mathbf{0}$.
- **Theorem 11**: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system $A\mathbf{x} = \mathbf{0}$ of m homogenous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Basis

- One such matrix is the nxn identity matrix. Its columns are denoted by $e_1,\dots,e_n$
- The set{e_1 , e_2, …, e_n} is called the standard basis for R^n

- **Definition**: A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

Theorem 12 - Basis for a Subspace

- **Theorem 12**: The pivot columns of a matrix A form a basis for the column space of A.
- adapt previous example
- Warning: in determining a basis for the column space of A, use the pivot columns of A not those of the row echelon form of A

Dimension and Rank

The Dimension of a Subspace

- The **dimension** of a nonzero subspace H, denoted dim H, is the number of vectors in any basis for H.
- The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A:

$$\text{rank } A = \dim \text{Col } A$$

- If someone asks you to determine the rank of the matrix…
  - Find the echelon form of A
  - Count the number of pivots, this is the rank

More Theorems

- **Theorem 13** If a matrix A has n columns, then

$$\dim \operatorname{Col} A + \dim \operatorname{Nul} A = n, \text{ or equivalently}$$

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

Theorem 14 Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

The Invertible Matrix Theorem Cont

- **The Invertible Theorem (continued)** Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

  m. The columns of A form a basis of $\mathbb{R}^n$.

  n. $\text{Col } A = \mathbb{R}^n$

  o. $\dim \text{Col } A = n$

  p. $\text{rank } A = n$

  q. $\text{Nul } A = \{\mathbf{0}\}$

  r. $\dim \text{Nul } A = 0$

Introduction to Determinants

Determinants

Definition: If A is a matrix with a row i and column j, then A_{ij} is the submatrix of A formed by deleting row i and column j

- **Definition**: For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm\, a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j}\, a_{1j} \det A_{1j}$$

- Let's see an example

Lead up to Theorem 1

To state the next theorem, it is convenient to write the definition of det A in a slightly different form. Given $A = [a_{ij}]$, the **(*i*, *j*)-cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij} \tag{4}$$

- Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

- This formula is called a **cofactor expansion across the first row** of A.

Theorem 1 and Theorem 2

Theorem 1: The determinant of an $n \times n$ matrix A can be computed by a cofactor across any row or down any column. The expansion across the i th row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

- **Theorem 2**: If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

Properties of Determinants

Properties of Determinants: Theorem 3

- **Theorem 3**: Let A be a square matrix
  a) If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
  b) If two rows of A are interchanged to produce B, then $\det B = -\det A$.
  c) If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$

Properties of Determinants: Theorem 4 and Theorem 5

Theorem 4: A square matrix A is invertible if and only if $\det A \neq 0$.

- **Theorem 5:** If A is a $n \times n$ matrix, then $\det A^T = \det A$.

Determinants and Matrix Products

- **Theorem 6:** If A and B are $n \times n$ matrices, then

$$\det AB = (\det A)(\det B).$$

Midterm 2 Process

About the Exam

I realize some of the processes are long and will take that into account in posing questions

- you maybe asked to do a partial computation – read carefully

- Make sure to know the definitions
- Make sure to know the statements of the theorems
- From midterm 1 - make sure to write the row operation if asked

Thank you for coming :)
Good Luck!

Link for the practise questions:
https://drive.google.com/file/d/1AqSuRcK1CG-EM98vTTmEKXuVA7ZDfuVZ/view?usp=sharing

Answers - some do not show all of the steps - like in row reduction:

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